

# The Chandrasekhar's Equation for Two-Dimensional Hypothetical White Dwarfs

Sanchari De and Somenath Chakrabarty<sup>†</sup>

Department of Physics, Visva-Bharati, Santiniketan-731235, India

<sup>†</sup>somenath.chakrabarty@visva-bharati.ac.in

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## Abstract

In this article we have extended the original work of Chandrasekhar on the structure of white dwarfs to the two-dimensional case. Although such two-dimensional stellar objects are hypothetical in nature, we strongly believe that the work presented in this article has some academic interest. In particular it is a 2-D version of Newtonian gravity. The electromagnetic or the coulomb problem in two dimension has already been investigated in greater detail, however, this particular problem with the same logarithmic type potential has not been studied before.

## 1 introduction

White dwarfs are the end product of intermediate massive and low mass stars. At the late stage of evolution of such stars, because of intense stellar wind, there are huge loss of matter from the outer part of such stellar objects. This is some kind of instability in this type of stars developed at the late stage of evolution. Most of the mass of these stars are thrown out and they form a gaseous nebula like structure, which can be observed by high power optical telescope. The compact object at the central region of this gaseous nebula is called the white dwarf. The mass of this compact object is very close to the solar mass with the size of a big planet (Jupiter like). Because of this very reason the gaseous region along with the massive central object is called planetary nebula. The formation of planetary nebula is a quite process compared to supernova explosion of massive stars and super-massive stars, which produce neutron star or in the extreme case, black holes [1, 2, 3].

Now depending on whether the mass of the progenitor is comparable with the solar mass or heavier than sun, inside the main-sequence stars the conversion of hydrogen to helium is going on continuously through either p-p chain reactions or CNO cycles [2, 3, 4]. To study the structure of such main-sequence stars a polytropic equation of state of the matter is considered. These stars are sometime also called main-sequence polytropic stars. The form of polytropic equation of state is given by [1, 2, 3, 4]

$$P = K\rho^\Gamma \tag{1}$$

where  $P$  is the kinetic pressure of the constituents,  $\rho$  is the mass density,  $K$  and  $\Gamma$  are constants. Both these constants depend on the physical properties of the matter. The constant  $\Gamma$  is called the polytropic index, which is more or less like the adiabatic index  $\gamma$ . Now the stellar matter inside the main-sequence stars are mainly a fully ionized hydrogen plasma. Further, the temperature of the matter is so high that the plasma is non-degenerate in nature. The temperature is also high enough for the hydrogen ions to overcome the inter-ionic coulomb barrier and undergo thermonuclear fusion reaction to form helium. The gravitational collapse of these stars are opposed by the kinetic pressure of the matter. To study the structure and gross properties of these main-sequence stars, the well known Lane-Emden equation is used [1, 2, 3, 4]. The Lane-Emden equation is essentially obtained by combining the hydrostatic equilibrium equation with the polytropic equation of state of the stellar matter.

On the other hand there is no thermo-nuclear reactions inside the white dwarf stars. The object is mainly made up of dense and fully ionized carbon or oxygen matter at the inner region with a crust of helium and a very thin layer of hydrogen gas at the skin of the star. The crustal matter is also fully ionized because of very high matter density. In this case the stability against gravitational collapse is governed by the degeneracy pressure of electron gas. Whereas the mass of the white dwarf star comes from the baryons, which are assumed to be at rest.

Therefore in general the equation of state of matter inside a white dwarf can not be represented by the polytropic form. In this case not the kinetic pressure, but the degeneracy pressure of the electron gas will contribute in the hydrostatic equilibrium condition. The form of the equation to study the structure and gross properties of white dwarf stars was originally derived by Chandrasekhar [4, 5], which coincides with the Lane-Emden equation only in some special situation.

In this article we have made an extension of Chandrasekhar's original work and investigated the gross properties of two dimensional white dwarf stars. Since there is no physical existence of such objects, therefore just for the sake of academic interest we have developed this formalism [4, 5]. To the best of our knowledge, this problem has not been reported before.

## 2 Basic Formalism

The surface density of degenerate electron gas is given by [6, 7]

$$n_e = \frac{N}{S} = \frac{p_F}{2\pi\hbar^2} \quad (2)$$

where  $p_F$  is the electron Fermi momentum. Hence the mass density

$$\rho = n_e \mu_e m_p \quad (3)$$

where  $m_p$  is the baryon mass and  $\mu_e$  is the electron mean molecular weight. It can very easily be shown that the expression for pressure of the degenerate electron gas in two-dimension is given by

$$P = \frac{c^2}{2\pi\hbar^2} \int_0^{p_F} \frac{p^3 dp}{(p^2 c^2 + m_e^2 c^4)^{1/2}} \quad (4)$$

where  $m_e$  is the electron rest mass. Defining  $y = p/m_e c$  and  $\xi = p_F/m_e c$ , we have

$$P = C f(\xi) \quad (5)$$

where

$$C = \frac{(m_e c^2)^3}{2\pi(\hbar c)^2} \quad (6)$$

and

$$f(\xi) = \int_0^\xi y^2 d(1 + y^2)^{1/2} \quad (7)$$

similarly we have for the mass density

$$\rho = C'\xi^2 \quad (8)$$

where

$$C' = \mu_e \frac{m_p(m_e c^2)^2}{2\pi(\hbar c)^2} \quad (9)$$

Now the hydrostatic equilibrium equation for white dwarf stars is given by [1, 4]

$$\frac{dP}{dr} = -g(r)\rho(r) \quad (10)$$

Now in two-dimension

$$g(r) = \frac{G}{r} \int_0^r 2\pi r' \rho(r') dr' \quad (11)$$

Substituting  $g(r)$  into eqn.(10) and then differentiating with respect to  $r$ , then after rearranging some of the terms, we have

$$\frac{1}{r} \frac{d}{dr} \left( \frac{r}{\rho} \frac{dP}{dr} \right) + 2\pi G \rho = 0 \quad (12)$$

One can start from this equation to obtain Lane-Emden equation in two dimension using polytropic equation of state. However, in this article we shall not discuss Lane-Emden equation. In some other communication we shall investigate the structure and gross properties of two-dimensional polytropic main-sequence stars using the Lane-Emden equation [8]. Now substituting  $P$  and  $\rho$  from eqn.(5) and eqn.(8) in eqn.(12), we get

$$\frac{C}{C'} \frac{1}{r} \frac{d}{dr} \left( \frac{r}{\xi} \frac{df}{dr} \right) + 2\pi G C' \xi^2 = 0 \quad (13)$$

Since

$$\frac{1}{\xi^2} \frac{df}{dr} = \frac{d}{dr} (1 + \xi^2)^{1/2} \quad (14)$$

we have after substituting  $x^2 = 1 + \xi^2$

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dx}{dr} \right) + \frac{2\pi G C'^2}{C} (x^2 - 1) = 0 \quad (15)$$

Therefore  $x = 1$  for  $\xi = 0$ , the extreme non-relativistic situation, whereas  $x \rightarrow \infty$ , the ultra-relativistic condition. Then Following [4], we substitute

$$U = \frac{x}{x_c} \quad \text{and} \quad Z = \frac{r}{A}$$

where  $x_c$  represents the value of  $x$  at the centre and  $A$  is an unknown constant. Therefore  $U = 1$  at the centre. Then we have

$$\frac{1}{Z} \frac{d}{dZ} \left( Z \frac{dU}{dZ} \right) + \frac{2\pi G C'^2 x_c}{C} A^2 \left( U^2 - \frac{1}{x_c^2} \right) = 0 \quad (16)$$

Choosing

$$A = \left( \frac{C}{2\pi G x_c} \right)^{1/2} \times \frac{1}{C'} \quad (17)$$

we have

$$\frac{1}{Z} \frac{d}{dZ} \left( Z \frac{dU}{dZ} \right) + \left( U^2 - \frac{1}{x_c^2} \right) = 0 \quad (18)$$

This is the differential equation describing the structure of white dwarfs in two-dimension, i.e., it is the modified version of Chandrasekhar equation. The corresponding three dimensional form, which was first obtained by Chandrasekhar, is given by [4, 5]

$$\frac{1}{Z^2} \frac{d}{dZ} \left( Z^2 \frac{dU}{dZ} \right) + \left( U^2 - \frac{1}{x_c^2} \right)^{3/2} = 0 \quad (19)$$

Just like the original version of Chandrasekhar's equation (eqn.(19)), the differential equation, given by eqn.(18) can not be solved analytically. To obtain the numerical solution for this second order differential equation, we use the following initial conditions At the centre,  $r = 0$ , i.e.,  $Z = 0$  and  $U = 1$ , which is the maximum value of  $U$ , therefore at the centre  $dU/dZ = 0$ . On the other hand, the surface of the white dwarf is obtained from the following condition. At  $Z = Z_s$ , the surface value,  $\rho = 0$ , therefore  $\xi_s = 0$ ,  $x_s = 1$  and  $U_s = 1/x_c$ . Then from eqn.(8), we can write down the expression for matter density in the following form

$$\rho = C' x_c^2 \left( U^2 - \frac{1}{x_c^2} \right) \quad (20)$$

The radius of the white dwarf star is given by

$$R = AZ_s = \left( \frac{C}{2\pi G x_c} \right)^{1/2} \times \frac{Z_s}{C'} \quad (21)$$

and the corresponding mass of the white dwarf can be obtained from the integral

$$M = 2\pi \int_0^R r dr \rho(r) \quad (22)$$

Using  $\rho(r)$  from eqn.(20) and changing the integration variable to  $Z$ , we have

$$M = -\frac{C x_c Z_s}{G C'} \frac{dU}{dZ} \Big|_{Z=Z_s} \quad (23)$$

In the evaluation of the mass  $M$  of the white dwarf, we use  $\mu_e = 2$ , assuming that there is no hydrogen in white dwarf star and the surface value of the gradient  $dU/dZ$  is obtained from the numerical solution of eqn.(18).

In fig.(1) we have shown the variation of mass of the white dwarf stars with  $x_c$ , or indirectly with the central density of the star. Whereas in fig.(2) the variation of radius of the star with  $x_c$  is shown. From the figures it is quite obvious that a white dwarf star in two-dimension becomes more compact in size but at the same time massive with the increase in central density. The mass of the star becomes infinitely large as the central density tending to infinity. In fig.(3) we have shown the mass-radius relation for such objects. We have chosen the upper limit of  $x_c$  in such a way that the mass of the object become  $\approx 1.41 M_\odot$  [9, 10].

### 3 Conclusion

In this article we have extended the original idea of Chandrasekhar. The formalism developed is for the two-dimensional white dwarf stars. Of course the objects are hypothetical in nature. Although in 2-D Newtonian gravity the potential has the same logarithmic nature as in the case of Coulomb problem, on which a lot of work has been done, there was no reported result on 2-D version of Chandrasekhar equation.

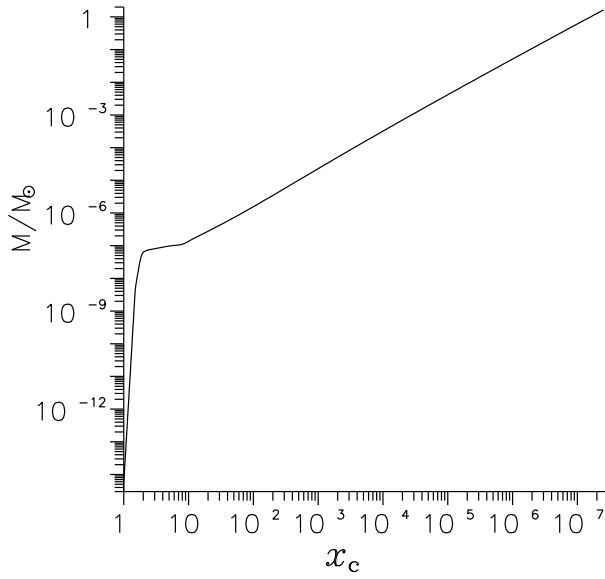


Figure 1: Variation of mass with  $x_c$

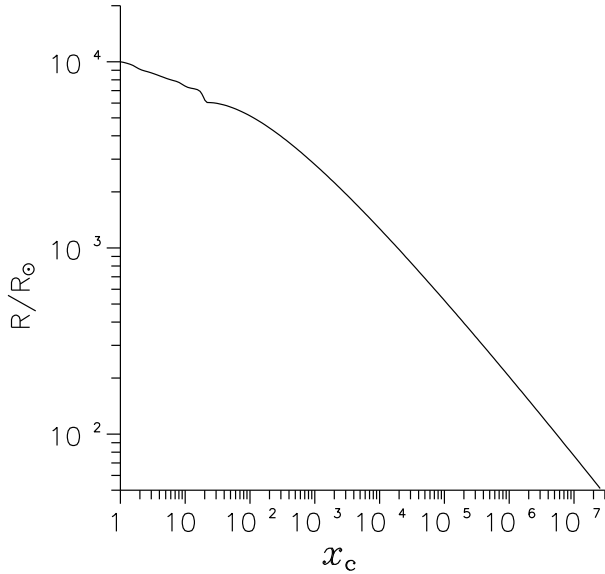


Figure 2: Variation of Radius with  $x_c$

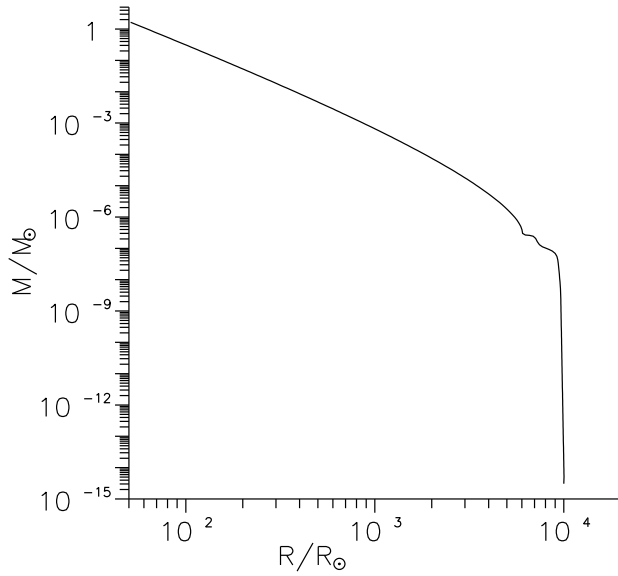


Figure 3: Mass-Radius relation

## References

- [1] S.L. Shapiro and S.A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, John Wiley and Sons, New York, (1983).
- [2] T. Padmanabhan, Theoretical Astrophysics, Vol. II: Star and Stellar Systems, Cambridge University Press, 2001.
- [3] A. Rai Choudhuri, Astrophysics for Physicists, Cambridge University Press, 2010.
- [4] H.Q. Huang and K.N. Yu, Stellar Astrophysics, Springer, (1998).
- [5] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, Univ. of Chicago Press, 1939..
- [6] L.D. Landau and E.M. Lifshitz, Statistical Physics, Part-I, Butterworth-Heimenann, Oxford, 1980.
- [7] K. Huang, Statistical Mechanics, Wiley-Eastern Pvt. Ltd., New Delhi, 1975.
- [8] Sanchari De and Somenath Chakrabarty, to be communicated.
- [9] T. Hamada and E.E. Salpeter, Astrophys. Jour., **134**, 683 (1961).
- [10] J.P. Cox and R.T. Giuli, Stellar Structure, Gordon and Breach Science Publishers Inc, 1968.